*RESEARCH INVENTY: International Journal of Engineering and Science*
*ISSN: 2278-4721, Vol. 1, Issue 7 (November 2012), PP 01-05*
*www.researchinventy.com*

# Operational Calculus of Canonical Cosine Transform

**[1], S. B. Chavhan, [2], V.C. Borkar**
Department of Mathematics & Statistics,
Yeshwant Mahavidyalaya, Nanded-431602 (India)

**Abstract:** This paper studies different properties of canonical cosine transform. Modulation heorem is also proved. We have proved some important results about the Kernel of canonical cosine transform.

**Keywords:** Generalized function, Canonical transform, Canonical Cosine transforms, Modulation theorem, Fourier transform.

## I. Introduction

The Fourier transform is certainly one of the best known of integral transforms, since its introduction by Fourier in early 1800's.

A much more general integral transform, namely, linear canonical transform was introduced in 1970 by Moshinsky and Quesne [5]. In 1980, Namias [6] introduced the fractional Fourier transform using eigen value and eigen functions.

Almeida [1], [2] had introduced it and proved many of its properties. Zayed [9] had discussed about product and convolution concerning that transform. Bhosale and Choudhary [3] had studies it as a tempered distribution.Number of applications of fractional Fourier transform in signal processing,image processing filtering optics, etc are studied. Pei and ding [7] ,[8] had studied linear canonical transforms.
The definition canonical cosine transform is as follows,

$$\{CCT\ f(t)\}(s) = \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\,dt \qquad \text{for } b \neq 0$$

$$= \sqrt{d}.e^{\frac{i}{2}cds^2} f(d.s) \qquad \text{for } b = 0$$

Notation and terminology of this paper is as per Zemanian [10], [11].The paper is organized as follows. Section 2 gives the definition of canonical cosine transform on the space of generalized function and states the property of the kernel of the canonical cosine transform. In section 3 modulation theorem is proved. In section 4, time reverse property, linear property, parity are proved. In section 5, operation transforms in terms of parameter are discussed and lastly the conclusion is stated.

## II. Generalized Canonical Cosine Transform

**2.1 Definition :** The canonical cosine transform of generalized function is defined as

$$\{CCT\ f(t)\}(s) = \langle f(t), K_{(a,b,c,d)}(t,s) \rangle$$

Where kernel $K_{(a,b,c,d)}(t,s) = \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right)$ ……..(1)

Clearly Kernel $K_{(a,b,c,d)}(t,s) \in E$ and $K_{(a,b,c,d)}(t,s) \in E(R^n)$ the kernel (1) Satisfies the following properties.

**2.2 Properties of Kernel :**

**2.3** $K_{(a,b,c,d)}(t,s) \neq K_{(a,b,c,d)}(s,t)$ if $a \neq d$

**2.4** $K_{(a,b,c,d)}(t,s) = K_{(a,b,c,d)}(s,t)$ if a = d

**2.5** $K_{(a,b,c,d)}(t,s) = K_{(a,-b,-c,d)}(t,s)$

**2.6** $K_{(a,b,c,d)}(-t,s) = K_{(a,b,c,d)}(t,-s)$

**2.7** Definition of canonical sine transforms as ,

$$\{CST\ f(t)\}(s) = -i\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\sin\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt \qquad \text{for } b \neq 0 \qquad \ldots\ldots(2)$$

$$= \sqrt{d}.e^{\frac{i}{2}cds^2} f(d.s) \qquad \text{for } b = 0$$

As we have obtained few properties of canonical sine transform in [4], we establish similar properties of canonical cosine transform here.

## III. Modulation Theorem for Canonical Cosine Transform

**Theorem 3.1:** If $\{CCT\ f(t)\}(s)$ is canonical cosine transform of f(t), $f(t)\in E^1(R^1)$ then

$$\{CCT\cos(ut)\,f(t)\}(s) = \frac{e^{\frac{i}{2}(du^2b)}}{2}\left[\{CCT\ f(t)\}(s+bu)e^{-idus} + \{CCT\ f(t)\}(s-bu)e^{idus}\right] \qquad \ldots\ldots.(3)$$

**Proof:** Using definition of canonical cosine transform

$$\{CCT\ f(t)\}(s) = \frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$$

$$\{CCT\cos(ut)\,f(t)\}(s) = \frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos(ut)\,f(t)dt$$

$$= \frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\frac{1}{2}\int_{-\infty}^{\infty}2\cos\left(\frac{s}{b}t\right)\cos(ut)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$$

$$= \frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\frac{1}{2}\int_{-\infty}^{\infty}\left(\cos\left(\frac{s}{b}+u\right)+\cos\left(\frac{s}{b}-u\right)\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{(s+ub)}{b}\right)t\,e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt + \frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s-ub}{b}\right)t\,e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\right]$$

$$= \frac{1}{2}\left\{\left[\frac{1}{\sqrt{2\pi ib}}d^{\frac{i}{2}\left(\frac{d}{b}\right)(s+ub)^2}e^{-i(dus)}e^{-\frac{i}{2}(du^2b)}\int_{-\infty}^{\infty}\cos\left(\frac{s+ub}{b}\right)t\,e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt\right]\right.$$
$$\left.+\left[\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-ub)^2}e^{iuds}e^{-\frac{i}{2}(du^2b)}\int_{-\infty}^{\infty}\cos\left(\frac{s-ub}{b}\right)t\,e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt\right]\right\}$$

$$= \frac{e^{\frac{i}{2}(du^2b)}}{2}\left[\{CCT\ f(t)\}(s+bu)e^{-idus} + \{CCT\ f(t)\}(s-bu)e^{idus}\right]$$

**Theorem 3.2:** If $\{CCT\ f(t)\}(s)$ is canonical cosine transform of $f(t), f(t)\in E^1(R^1)$

Then $\{CCT\sin ut\ f(t)\}(s) = \frac{-i}{2}e^{\frac{i}{2}(du^2b)}\left[\{CST\ f(t)\}(s-ub)e^{idsu} + \{CST\ f(t)\}(s+ub)e^{-idsu}\right]$ ……(4)

**Proof :** By using definition canonical cosine transform

$$\{CCT\ f(t)\}(s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{\infty}^{\infty}\cos\left(\frac{S}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t)dt$$

$$\{CCT\sin ut\ f(t)\}(s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right).e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\sin ut\ f(t)dt$$

$$=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\frac{1}{2}\int_{-\infty}^{\infty}\left(2\cos\left(\frac{s}{b}t\right)\sin ut\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}f(t)dt$$

$$=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\frac{1}{2}\int_{-\infty}^{\infty}\left(\sin\left(\frac{s}{b}t+ut\right)-\sin\left(\frac{s}{b}t-ut\right)\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t)dt$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\sin\left(\frac{s+ub}{b}\right)t\,e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t)dt-\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\sin\left(\frac{s-ub}{b}\right)t\,e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}f(t)dt\right]$$

$$=\frac{-ie^{-\frac{i}{2}\left(du^2b\right)}}{2}\left[\{CST\ f(t)\}(s-bu)e^{idsu}+\{CST\ f(t)\}(s+ub)e^{-idus}\right]$$

**Theorem 3.3:** If $\{CCT\ f(t)\}(s)$ is canonical cosine transform of $f(t), f(t)\in E^1(R^1)$

Then

$$\{CCT\,e^{iut}f(t)\}(s)$$
$$=\frac{i}{2}e^{\frac{i}{2}\left(du^2b\right)}\left\{e^{-idsu}\left[\{CCT\ f(t)\}(s+ub)-\{CST\ f(t)\}(s+ub)\right]+e^{idus}\left[\{CCT\ f(t)\}(s-ub)-\{CST\,f(t)\}(s-ub)\right]\right\} \quad \text{..... (5)}$$

## IV. Some useful Properties:

In this section we discuss some useful results.

**4.1 Time Reverse Property:** If $\{CCT\ f(t)\}$ is canonical cosine transforms of $f(t)$, $f(t)\in E^1(R^1)$ then

$$\{CCT\ f(-t)\}(s)=\{CCT\ f(t)\}(s) \quad \text{.......... (6)}$$

**Proof :** Using definition of canonical cosine transforms of $f(t)$

$$\{CCT\ f(t)\}(s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}f(t)dt$$

$$\{CCT\ f(-t)\}(s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}f(-t)dt$$

Put $-t=x$ $\quad\therefore t=-x$

Hence $dt=-dx,$ also $t\to-\infty\ to\infty,\ x\to\infty,-\infty$

$$\therefore\{CCT\ f(-t)\}(s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(-\frac{s}{b}x\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)\left(-x^2\right)}f(x)(-dx)$$

$$\{CCT\ f(-t)\}(s) = -\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{+\infty}^{-\infty} \cos\left(\frac{s}{b}x\right) e^{\frac{i}{2}\left(\frac{d}{b}\right)x^2} f(x).dx$$

Replacing $x$ by $t$ again

$$= \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2} f(t)dt$$

$$\therefore \{CCT\ f(t)\}(s) = \{CCT\ f(t)\}(s)$$

**4.2 Parity:** If $\{CCT\ f(t)\}(s)$ is canonical cosine transform of $f(t) \in E^1(R^1)$ then

$$\{CCT\ f(-t)\}(s) = \{CCT\ f(t)\}(-s) \qquad \text{........ (7)}$$

**4.3 Linearity Property:** If $C_1, C_2$ are constant and $f_1,\ f_2$ are functions of $t$ then

$$\{CCT[C_1 f_1(t) + C_2 f_2(t)]\}(s) = C_1\{CCT\ f_1(t)\}(s) + C_2\{CCT f_2(t)\}(s) \qquad \text{...... (8)}$$

## V. Operation Transform in Terms of Parameters

This section presents the operational formulae for canonical cosine transform with shifted parameter '*s* ' and differential of canonical cosine transform with respect to *s*.

**Theorem 5.1 :** If $\{CCT\ f(t)\}(s)$ is canonical cosine transform of $f(t)$ in terms of parameter s, then

$$\{CCT\ f(t)\}(s+\alpha) = e^{\frac{i}{2}\left(\frac{d}{b}\right)(2s\alpha+\alpha^2)} \left[\left\{CCT\cos\left(\frac{\alpha}{b}t\right) f(t)\right\}(s) + \frac{1}{i}\left\{CST\sin\left(\frac{\alpha}{b}t\right) f(t)\right\}(s)\right] \qquad \text{............ (9)}$$

**Proof:** We know that' $\{CCT\ f(t)\}(s) = \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)S^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$

$$\{CCT\ f(t)\}(s+\alpha) = \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)(S+\alpha)^2} \int_{-\infty}^{\infty} \cos\frac{(s+\alpha)}{b} t + e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$$

$$= \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s^2+2s\alpha+\alpha^2)} \int_{-\infty}^{\infty} \left\{\cos\left(\frac{s}{b}t\right)\cos\left(\frac{\alpha}{b}t\right) - \sin\left(\frac{s}{b}t\right)\sin\left(\frac{\alpha}{b}t\right)\right\} e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2} f(t)dt$$

$$= \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s^2+2s\alpha+\alpha^2)} \left[\int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right)\cos\left(\frac{\alpha}{b}t\right) e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2} f(t)dt - \int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{\alpha}{b}t\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt\right]$$

$$= \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(2s\alpha+\alpha^2)} \left[\int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right).\cos\left(\frac{\alpha}{b}t\right).e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt - \int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{\alpha}{b}t\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt\right]$$

$$= e^{\frac{i}{2}\left(\frac{d}{b}\right)(2s\alpha+\alpha^2)} \left[\left\{CCT\cos\left(\frac{\alpha}{b}t\right) f(t)\right\}(s) + \frac{1}{i}\left\{CST\sin\left(\frac{\alpha}{b}t\right) f(t)(s)\right\}\right]$$

**Theorem 5.2:** If $\{CCT\ f(t)\}(s)$ is canonical cosine transform of f(t) in terms of parameter S

Then $$\frac{d}{ds}\left[CCT\ f(t)\right](s)=i\left(\frac{ds}{b}\right)\left\{CCT\ f(t)\right\}(s)-i\left\{CST\left(\frac{t}{b}\right)f(t)\right\}(s)$$ ……….. (10)

Or $$\frac{d}{ds}\left[CCT\ f(t)\right](s)=i\left[\left(\frac{ds}{b}\right)\left\{CCT\ f(t)\right\}(s)-\left\{CST\left(\frac{t}{b}\right)f(t)\right\}(s)\right]$$ ………… (11)

**Proof:** Using definition of canonical cosine transform

$$\left\{CCT\ f(t)\right\}(s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t)dt$$

$$\therefore\frac{d}{ds}\left\{CCT\ f(t)\right\}(s)=\frac{d}{ds}\left[\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t)dt\right]$$

$$=\frac{1}{\sqrt{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left\{\frac{d}{ds}\left[e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\right]f(t)dt\right\}$$

$$=\frac{1}{\sqrt{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left[e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)i\left(\frac{ds}{b}\right)-\sin\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)s^2}\left(\frac{t}{b}\right)\right]f(t)dt$$

$$=i\left(\frac{ds}{b}\right)\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}f(t)dt-\frac{i}{i}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\sin\left(\frac{s}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\left(\frac{t}{b}\right)f(t)dt$$

$$=i\left(\frac{ds}{b}\right).\left\{CCT\ f(t)\right\}(s)+\frac{1}{i}\left\{CST\left(\frac{t}{b}\right)f(t)\right\}(s)$$

$$=\left\{CST\left(\frac{t}{b}\right)f(t)\right\}(s)-\frac{ds}{b}\left\{CCT\ f(t)\right\}(s)$$

$$\therefore\frac{d}{ds}\left\{CCT\ f(t)\right\}(s)==i\left(\frac{ds}{b}\right).\left\{CCT\ f(t)\right\}(s)+\frac{1}{i}\left\{CST\left(\frac{t}{b}\right)f(t)\right\}(s)$$

$$\frac{d}{ds}\left\{CCT\ f(t)\right\}(s)=\left\{CST\ f(t).\left(\frac{t}{b}\right)\right\}(s)-\left(\frac{ds}{d}\right)\left\{CCT\ f(t)\right\}(s)$$

## VI. Conclusion:

The canonical cosine transform, which is the generalization of number of transforms is itself generalized to the spaces of generalized functions as per Zemanian. This transform is used to solve ordinary or partial differential equations. This transform is an important tool in signal processing and many other branches of engineering. In this paper we have discussed some of its properties.

## References:


[1] Almeida L.B., "An introduction to the angular Fourier transform", IEEE, 257-260 (1993).

[2] Almeida L.B., "The fractional Fourier transform and time frequency representation IEEE trans on signal processing",Vol. 42, No.11 (1994).

[3] Bhosale B.N. and Choudhary M.S.,"Fractional fourier transform of distributions of compact support", Bull. Cal. Math. Soc., Vol. 94, No.5, 349-358 (2002).

[4] Chavhan S.B., Borkar V.C.,"Canonical sine transform and their unitary representation", International Journal of Contemporary Mathematical Sciences. (Accepted).

[5] Moshinky M. Quesne ., "Linear canonical transform and their unitary representation", J. Math, Phy., Vol.12, (No.), 1772-1783 (1971).

[6] Namias V., "The fractional order Fourier transform and its applications in quantum mechanics", I. Inst. Maths. appli., 25, 241-265 (1980).

[7] Peis, Ding J., "Eigen function of linear canonical transform", IEEE, trans. sign. proc. Vol.50, No.1 (2002).

[8] Peis, Ding J., "Fractional cosine, sine and Hartley transform, trans. on signal processing", Vol.50, No.7, Pub. P.1661-1680 (2002).

[9] Zayed A., "A convolution and product theorem for the fractional Fourier transform", IEEE sign. proc. letters Vol.5, No.4, 101-103 (1998).

[10] Zemanian A.H., "Distribution theory and transform analysis", McGraw Hill, New York, (1965).

[11] Zemanian A.H., "Generalized Integral Transform", Inter Science Publisher's New York, (1968).