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Homogeneous Quadratic Equation With Three Unknowns $z^2 = 11x^2 - 7y^2$

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ABSTRACT:
In this paper, different sets of non-zero distinct integer solutions to the homogeneous quadratic equation with three unknowns given by $z^2 = 11x^2 - 7y^2$ are obtained. A few interesting relations among the solutions are presented.
KEYWORDS: *Homogeneous quadratic, Quadratic with three unknowns, Integer solutions.*

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I. INTRODUCTION

Ternary quadratic equations are rich in variety [1- 4, 17-20].For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, we consider yet another interesting homogeneous ternary quadratic equation $z^2 = 11x^2 - 7y^2$ and obtain infinitely many non-trivial integral solutions.

II. METHOD OF ANALYSIS

The homogeneous quadratic equation with three unknowns to be solved is

$$z^2 = 11x^2 - 7y^2 \tag{1}$$

Different sets of solutions in integers to (1) are illustrated below:

1.1 Way

Introduction of the linear transformations

$$x = X + 7T,\ y = X + 11T,\ z = 2W,\ (X \neq W \neq T) \tag{2}$$

in (1) leads to

$$X^2 - W^2 = 77T^2 \tag{3}$$

which is (3) is satisfied by

$$\begin{aligned} T &= 2rs \\ W &= 77r^2 - s^2 \\ X &= 77r^2 + s^2 \end{aligned} \tag{4}$$

Substituting the above values of X, W and T in (2), the non-zero distinct integral values of x ,y and z are given by

$$\left.\begin{aligned} x &= x(r,s) = 77r^2 + 14rs + s^2 \\ y &= y(r,s) = 77r^2 + 22rs + s^2 \\ z &= z(r,s) = 154r^2 - 2s^2 \end{aligned}\right\} \tag{5}$$

Thus (5) represent the non-zero integer solutions to (1).

1.2 Way

Write (3) as the system of double equations as shown in Table 1 below:

Table 1: SYSTEM OF DOUBLE EQUATIONS

SYSTEM	1	2	3	4	5	6	7	8	9	10	11	12
$X+W$	T^2	$7T^2$	$11T^2$	$77T^2$	$77T$	$11T$	$7T$	T	1	7	11	77
$X-W$	77	11	7	1	T	$7T$	$11T$	$77T$	$77T^2$	$11T^2$	$7T^2$	T^2

Solving each of the system of equations in Table 1, the corresponding values of X, W and T are obtained. Substituting the values of X, W and T in (2), the respective values of x, y and z are determined. The integer solutions to (1) obtained through solving each of the above system of equations are exhibited.

Solutions through SYSTEM 1:

$$x = 2k^2 + 16k + 46$$
$$y = 2k^2 + 24k + 50$$
$$z = 4k^2 + 4k - 76$$

Solutions through SYSTEM 2:

$$x = 14k^2 + 28k + 16$$
$$y = 14k^2 + 36k + 20$$
$$z = 28k^2 + 28k - 4$$

Solutions through SYSTEM 3:

$$x = 22k^2 + 36k + 16$$
$$y = 22k^2 + 44k + 20$$
$$z = 44k^2 + 44k + 4$$

Solutions through SYSTEM 4:

$$x = 154k^2 + 168k + 46$$
$$y = 154k^2 + 176k + 50$$
$$z = 308k^2 + 308k + 76$$

Solutions through SYSTEM 5:

$$x = 46T, y = 50T, z = 76T$$

Solutions through SYSTEM 6:

$$x = 16T, y = 20T, z = 4T$$

Solutions through SYSTEM 7:

$$x = 16T, y = 20T, z = -4T$$

Solutions through SYSTEM 8:

$$x = 46T, y = 50T, z = -76T$$

Solutions through SYSTEM 9:

$$x = 154k^2 + 168k + 46$$
$$y = 154k^2 + 176k + 50$$
$$z = -308k^2 - 308k - 76$$

Solutions through SYSTEM 10:

$$x = 22k^2 + 36k + 16$$
$$y = 22k^2 + 44k + 20$$
$$z = -44k^2 - 44k - 4$$

Solutions through SYSTEM 11:

$$x = 14k^2 + 28k + 16$$
$$y = 14k^2 + 36k + 20$$
$$z = -28k^2 - 28k + 4$$

Solutions through SYSTEM 12:

$$x = 2k^2 + 16k + 46$$
$$y = 2k^2 + 24k + 50$$
$$z = -4k^2 - 4k + 76$$

1.3 Way

Rewrite (3) as

$$X^2 - W^2 = 7 * 11 * T^2 \tag{6}$$

Write (6) in the form of ratio as

$$\frac{X + W}{11T} = \frac{7T}{X - W} = \frac{\alpha}{\beta}, (\beta \neq 0) \tag{7}$$

which is equivalent to the following two equations

$$\beta X + \beta W - 11T\alpha = 0$$
$$-X\alpha + W\alpha + 7T\beta = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} X &= 11\alpha^2 + 7\beta^2 \\ W &= 11\alpha^2 - 7\beta^2 \end{aligned}\right\} \tag{8}$$

$$T = 2\alpha\beta \tag{9}$$

Using (8) and (9) in (2), the non-zero distinct integral values of x, y and z are given by

$$\left.\begin{aligned} x(\alpha,\beta) &= 11\alpha^2 + 14\alpha\beta + 7\beta^2 \\ y(\alpha,\beta) &= 11\alpha^2 + 22\alpha\beta + 7\beta^2 \\ z(\alpha,\beta) &= 22\alpha^2 - 14\beta^2 \end{aligned}\right\} \tag{10}$$

Thus (10) represent the non-zero integer solutions to (1).

1.4 Way

Write (6) in the form of ratio as

$$\frac{X + W}{7T} = \frac{11T}{X - W} = \frac{\alpha}{\beta}, (\beta \neq 0) \tag{11}$$

which is equivalent to the following two equations

$$\beta X + \beta W - 7T\alpha = 0$$
$$-X\alpha + W\alpha + 11T\beta = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} X &= 7\alpha^2 + 11\beta^2 \\ W &= 7\alpha^2 - 11\beta^2 \end{aligned}\right\} \tag{12}$$

Using (9) and (12) in (2), the non-zero distinct integral values of x, y and z are given by

$$\left.\begin{aligned} x(\alpha,\beta) &= 7\alpha^2 + 14\alpha\beta + 11\beta^2 \\ y(\alpha,\beta) &= 7\alpha^2 + 22\alpha\beta + 11\beta^2 \\ z(\alpha,\beta) &= 14\alpha^2 - 22\beta^2 \end{aligned}\right\} \tag{13}$$

Thus (13) represent the non-zero integer solutions to (1).

1.5 Way

Write (6) in the form of ratio as

$$\frac{X - W}{11T} = \frac{7T}{X + W} = \frac{\alpha}{\beta}, (\beta \neq 0) \tag{14}$$

which is equivalent to the following two equations

$$\beta X - \beta W - 11T\alpha = 0$$
$$-X\alpha - W\alpha + 7T\beta = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} X &= -11\alpha^2 - 7\beta^2 \\ W &= 11\alpha^2 - 7\beta^2 \end{aligned}\right\} \tag{15}$$

$$T = -2\alpha\beta \tag{16}$$

Using (15) and (16) in (2), the non-zero distinct integral values of x, y and z are given by

$$\left.\begin{aligned} x(\alpha,\beta) &= -11\alpha^2 - 14\alpha\beta - 7\beta^2 \\ y(\alpha,\beta) &= -11\alpha^2 - 22\alpha\beta - 7\beta^2 \\ z(\alpha,\beta) &= 22\alpha^2 - 14\beta^2 \end{aligned}\right\} \quad (17)$$

Thus (17) represent the non-zero integer solutions to (1).

1.6 Way

Write (6) in the form of ratio as

$$\frac{X-W}{7T} = \frac{11T}{X+W} = \frac{\alpha}{\beta}, (\beta \neq 0) \quad (18)$$

Which is equivalent to the following two equations

$$\beta X - \beta W - 7T\alpha = 0$$

$$-X\alpha - W\alpha + 11T\beta = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} X &= -7\alpha^2 - 11\beta^2 \\ W &= 7\alpha^2 - 11\beta^2 \end{aligned}\right\} \quad (19)$$

Using (16) and (19) in (2), the non-zero distinct integral values of x, y and z are given by

$$\left.\begin{aligned} x(\alpha,\beta) &= -7\alpha^2 - 14\alpha\beta - 11\beta^2 \\ y(\alpha,\beta) &= -7\alpha^2 - 22\alpha\beta - 11\beta^2 \\ z(\alpha,\beta) &= 14\alpha^2 - 22\beta^2 \end{aligned}\right\} \quad (20)$$

Thus (20) represent the non-zero integer solutions to (1).

1.7 Way

Rewrite (3) as

$$W^2 + 77T^2 = X^2 * 1 \quad (21)$$

Let

$$X = a^2 + 77b^2 \quad (22)$$

where a and b are non-zero integers.

Consider 1 as

$$1 = \left(\frac{(2 + i\sqrt{77})(2 - i\sqrt{77})}{81}\right) \quad (23)$$

Using (22), (23) in (21) and applying the method of factorization, define

$$(W + i\sqrt{77}T) = (a + i\sqrt{77}b)^2 \left(\frac{2 + i\sqrt{77}}{9}\right) \quad (24)$$

from which we have

$$\left.\begin{aligned} W &= \frac{1}{9}(2a^2 - 154ab - 154b^2) \\ T &= \frac{1}{9}(a^2 + 4ab - 77b^2) \end{aligned}\right\} \quad (25)$$

Using (22) and (25) in (2), the values of x, y and z are given by

$$\left.\begin{aligned} x &= \frac{1}{9}(16a^2 + 28ab + 154b^2) \\ y &= \frac{1}{9}(20a^2 + 44ab - 154b^2) \\ z &= \frac{1}{9}(4a^2 - 308ab - 308b^2) \end{aligned}\right\} \quad (26)$$

Since our interest is on finding integer solutions, replacing a by 3A, b by 3B in (26), the corresponding integer solutions to (1) are given by

$$\left.\begin{array}{l} x = x(A,B) = 16A^2 + 28AB + 154B^2 \\ y = y(A,B) = 20A^2 + 44AB - 154B^2 \\ z = z(A,B) = 4A^2 - 308AB - 308B^2 \end{array}\right\} \quad (27)$$

1.8 Way

Rewrite (3) as

$$X^2 - 77T^2 = W^2 * 1 \quad (28)$$

Let

$$W = a^2 - 77b^2 \quad (29)$$

where a and b are non-zero integers.

Take 1 as

$$1 = \left(\frac{(9+\sqrt{77})(9-\sqrt{77})}{4} \right) \quad (30)$$

Using (29), (30) in (28) and applying the method of factorization, define

$$(X + \sqrt{77}T) = (a + \sqrt{77}b)^2 \left(\frac{9+\sqrt{77}}{2} \right) \quad (31)$$

from which we have

$$\left.\begin{array}{l} X = \frac{1}{2}(9a^2 + 154ab + 693b^2) \\ T = \frac{1}{2}(a^2 + 18ab + 77b^2) \end{array}\right\} \quad (32)$$

Using (29) and (32) in (2), the values of x, y and z are given by

$$\left.\begin{array}{l} x(a,b) = 8a^2 + 140ab + 616b^2 \\ y(a,b) = 10a^2 + 176ab + 770b^2 \\ z(a,b) = 2a^2 - 154b^2 \end{array}\right\} \quad (33)$$

Thus (33) represent the non-zero integer solutions to (1).

1.9 Way

Rewrite (1) as

$$z^2 + 7y^2 = 11x^2 \quad (34)$$

Let

$$x = a^2 + 7b^2 \quad (35)$$

where a and b are non-zero integers.

Write 11 as

$$11 = (2 + i\sqrt{7})(2 - i\sqrt{7}) \quad (36)$$

Using (35), (36) in (34) and applying the method of factorization, define

$$(z + i\sqrt{7}y) = (a + i\sqrt{7}b)^2 (2 + i\sqrt{7}) \quad (37)$$

from which we have

$$\left.\begin{array}{l} y = a^2 + 4ab - 7b^2 \\ z = 2a^2 - 14ab - 14b^2 \end{array}\right\} \quad (38)$$

Thus (35) and (38) represent the non-zero integer solutions to (1).

1.10 Way

Write 11 as

$$11 = \frac{(13 + i\sqrt{7})(13 - i\sqrt{7})}{16} \quad (39)$$

Using (35), (39) in (34) and applying the method of factorization, define

$$(z + i\sqrt{7}y) = (a + i\sqrt{7}b)^2 \frac{(13 + i\sqrt{7})}{4} \quad (40)$$

from which we have

$$\left.\begin{aligned} y &= \frac{1}{4}\left(a^2 + 26ab - 7b^2\right) \\ z &= \frac{1}{4}\left(13a^2 - 14ab - 91b^2\right) \end{aligned}\right\} \quad (41)$$

Since our interest is on finding integer solutions, replacing a by 2A, b by 2B in (35) and (41), the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x &= x(A,B) = 4A^2 + 28B^2 \\ y &= y(A,B) = A^2 + 26AB - 7B^2 \\ z &= z(A,B) = 13A^2 - 14AB - 91B^2 \end{aligned}\right\} \quad (42)$$

1.11 Way

Write 11 as

$$11 = \frac{\left(23 + i5\sqrt{7}\right)\left(23 - i5\sqrt{7}\right)}{64} \quad (43)$$

Using (35), (43) in (34) and applying the method of factorization, define

$$\left(z + i\sqrt{7}y\right) = \left(a + i\sqrt{7}b\right)^2 \frac{\left(23 + i5\sqrt{7}\right)}{8} \quad (44)$$

from which we have

$$\left.\begin{aligned} y &= \frac{1}{8}\left(5a^2 + 46ab - 35b^2\right) \\ z &= \frac{1}{8}\left(23a^2 - 70ab - 161b^2\right) \end{aligned}\right\} \quad (45)$$

Since our interest is on finding integer solutions, replacing a by 4A, b by 4B in (35) and (45), the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x &= x(A,B) = 16A^2 + 112B^2 \\ y &= y(A,B) = 10A^2 + 92AB - 70B^2 \\ z &= z(A,B) = 46A^2 - 140AB - 322B^2 \end{aligned}\right\} \quad (46)$$

1.12 Way

Rewrite (1) as

$$7y^2 = 11x^2 - z^2 \quad (47)$$

Let

$$y = 11a^2 - b^2 \quad (48)$$

where a and b are non-zero integers.

Write 7 as

$$7 = \left(\sqrt{11} + 2\right)\left(\sqrt{11} - 2\right) \quad (49)$$

Using (48), (49) in (47) and applying the method of factorization, define

$$\left(\sqrt{11}x + z\right) = \left(\sqrt{11}a + b\right)^2\left(\sqrt{11} + 2\right) \quad (50)$$

from which we have

$$\left.\begin{aligned} x &= 11a^2 + 4ab + b^2 \\ z &= 22a^2 + 22ab + 2b^2 \end{aligned}\right\} \quad (51)$$

Thus (48) and (51) represent the non-zero integer solutions to (1).

1.13 Way

Write 7 as

$$7 = \left(4\sqrt{11} + 13\right)\left(4\sqrt{11} - 13\right) \quad (52)$$

Using (48), (52) in (47) and applying the method of factorization, define

$$\left(\sqrt{11}x + z\right) = \left(\sqrt{11}a + b\right)^2\left(4\sqrt{11} + 13\right) \quad (53)$$

from which we have

$$\left.\begin{aligned} x &= 44a^2 + 26ab + 4b^2 \\ z &= 143a^2 + 88ab + 13b^2 \end{aligned}\right\} \quad (54)$$

Thus (48) and (54) represent the non-zero integer solutions to (1).

1.14 Way

Write 7 as

$$7 = \frac{(4\sqrt{11}+1)(4\sqrt{11}-1)}{25} \tag{55}$$

Using (48), (55) in (47) and applying the method of factorization, define

$$(\sqrt{11}\,x + z) = (\sqrt{11}\,a + b)^2\,\frac{(4\sqrt{11}+1)}{5} \tag{56}$$

from which we have

$$\left.\begin{aligned} x &= \frac{1}{5}(44a^2 + 2ab + 4b^2) \\ z &= \frac{1}{5}(11a^2 + 88ab + b^2) \end{aligned}\right\} \tag{57}$$

Since our interest is on finding integer solutions, replacing a by 5A, b by 5B in (48) and (57), the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x &= x(A,B) = 220A^2 + 10AB + 20B^2 \\ y &= y(A,B) = 275A^2 - 25B^2 \\ z &= z(A,B) = 55A^2 + 440AB + 5B^2 \end{aligned}\right\} \tag{58}$$

1.15 Generation of solutions

A formula for generating sequence of solutions to the considered equation based on its given solution is illustrated.

Case (i)

Let (x_0, y_0, z_0) be a given solution to (1).

Let

$$x_1 = h - 2x_0,\ y_1 = h + 2y_0,\ z_1 = 2z_0 \tag{59}$$

be the second solution to (1), where h is a non-zero constant to be determined.

Substituting (59) in (1) and simplifying, one gets

$$h = 11x_0 + 7y_0$$

and thus

$$x_1 = 9x_0 + 7y_0,\ y_1 = 11x_0 + 9y_0$$

which is written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

Where $M = \begin{pmatrix} 9 & 7 \\ 11 & 9 \end{pmatrix}$ and t is the transpose.

Repeating the above process, the general solution (x_n, y_n) to (1) is given by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t,\ n = 1,2,3....$$

where $M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$

in which α, β are the eigen values of M and I is the unit matrix of order 2.

Now, it is seen after performing a few calculations,

$$\alpha = 9 + \sqrt{77},\ \beta = 9 - \sqrt{77}$$

Hence, we get

$$(x_n, y_n)^t = M^n (x_0, y_0)^t$$

where $M^n = \begin{pmatrix} Y_n & \frac{7}{2}X_n \\ \frac{11}{2}X_n & Y_n \end{pmatrix},\ Y_n = \frac{\alpha^n + \beta^n}{2},\ X_n = \frac{\alpha^n - \beta^n}{\sqrt{77}},\ n = 1,2,3......$

Taking $n = 1,2,3....$ in the above equation, one generates sequence of integer solutions based on the given solution.

Case (ii)

Let (x_0, y_0, z_0) be a given solution to (1).

Let

$$x_1 = 4x_0, y_1 = h - 4y_0, z_1 = h - 4z_0 \qquad (60)$$

be the second solution to (1), where h is a non-zero constant to be determined. Substituting (60) in (1) and simplifying, one gets

$$h = 7y_0 + z_0$$

and thus

$$y_1 = 3y_0 + z_0, z_1 = 7y_0 - 3z_0$$

which is written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

Where $M = \begin{pmatrix} 3 & 1 \\ 7 & -3 \end{pmatrix}$ and t is the transpose.

Repeating the above process, the general solution (y_n, z_n) to (1) is given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t, n = 1,2,3....$$

where $M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$

in which α, β are the eigen values of M and I is the unit matrix of order 2.

Now, it is seen after performing a few calculations,

$$\alpha = 4, \beta = -4$$

Hence, we get

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

where $M^n = \begin{pmatrix} (7\alpha^n + \beta^n)Y_n & X_n \\ 7X_n & (\alpha^n + 7\beta^n)Y_n \end{pmatrix}$, $Y_n = \frac{1}{8}$, $X_n = \frac{\alpha^n - \beta^n}{8}$, $n = 1,2,3......$

Taking $n = 1,2,3....$ in the above equation, one generates sequence of integer solutions based on the given solution.

Case (iii)

Let (x_0, y_0, z_0) be a given solution to (1).

Let

$$x_1 = h - 5x_0, y_1 = 5y_0, z_1 = h + 5z_0 \qquad (61)$$

be the second solution to (1), where h is a non-zero constant to be determined. Substituting (61) in (1) and simplifying, one gets

$$h = 11x_0 + z_0$$

and thus

$$x_1 = 6x_0 + z_0, z_1 = 11x_0 + 6z_0$$

which is written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

Where $M = \begin{pmatrix} 6 & 1 \\ 11 & 6 \end{pmatrix}$ and t is the transpose.

Repeating the above process, the general solution (x_n, z_n) to (1) is given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t, n = 1,2,3....$$

where $M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$

in which α, β are the eigen values of M and I is the unit matrix of order 2.

Now, it is seen after performing a few calculations,

$$\alpha = 6 + \sqrt{11}, \beta = 6 - \sqrt{11}$$

Hence, we get

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

where $M^n = \begin{pmatrix} Y_n & \frac{1}{2} X_n \\ \frac{11}{2} X_n & Y_n \end{pmatrix}$, $Y_n = (\alpha^n + \beta^n), X_n = \frac{\alpha^n - \beta^n}{\sqrt{11}}, n = 1,2,3 ......$

Taking $n = 1,2,3....$ in the above equation, one generates sequence of integer solutions based on the given solution.

III. CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the homogeneous quadratic diophantine equation. As the quadratic equations are rich in variety, one may search for other forms of quadratic equation with variables greater than or equal to three and obtain their corresponding observations.

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