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The Deficient C¹ Quartic Spline Interpolation

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ABSTRACT: *In this paper we have obtained the best error bounds for deficient Quartic Spline interpolation matching the given functional value at mid points and its derivative with two interior points with boundary condition.*

Key Words : *Error Bounds, Deficient, Quartic Spline, Interpolation, best error.*

1. INTRODUCTION

The most popular choice of reasonably efficient approximating functions still continues to be in favour of Quartic and higher degree (See deBoor [1]). In the study of Piecewise linear functions it is a disadvantage that we get corner's at joints of two linear pieces and therefore to achieve prescribed accuracy more data than higher order method are needed. Various aspects of cubic interpolatory splines have been extended number of authors, Dubean [3], Rana and Dubey [10]. Morken and Reimers [6]. Kopotum [5] has shown the equivalence of moduli of smoothness and application of univariate splines.Rana and Dubey [8] generalized result of Garry and Howell [4] for quartic spline interpolation. Rana, Dubey and Gupta [9] have obtained a precise error estimate concerning quartic spline interpolation matching the given functional value at intermediate points between successive mesh points and some boundary conditions.

II. EXISTENCE AND UNIQUENESS.

Let a mesh on [0, 1] be given by $P=\{0=x_0<x_1<-<x_n=1\}$ with $h_i=x_i-x_{i-1}$ for i=1, 2,.....n. Let L_k denote the set of all algebraic polynomials of degree not greater than K and s_i is the restrictions of s on $[x_{1-i},x_i]$ the class S (4,2, P) of deficient quartic spline of deficiency 2 is defined by

$$S(4,2,P)=\{s : s\in C^1[0,1],\ s_i\in L_k\ \ for\ i=1,2,...,n.\}$$

where in S* (4,2, P) denotes the class of all deficient quartic spline S (4,2, P) which satisfies the boundary conditions.

$$s(x_0)=f(x_0), s(x_n)=f(x_n) \qquad ....(2.1)$$

We introduced following problem.

2.1 PROBLEM

Suppose f ' exist over p, there under what restriction on h_i there exist a unique spline interpolation $s\in S^*(4,2,P)$, of f which satisfies the interpolating condition.

$$\left.\begin{array}{lcl} s(\beta_i) & = & f(\beta_i) \\ s'(\alpha_i) & = & f'(\alpha_i) \\ s'(\gamma_i) & = & f'(\gamma_i) \end{array}\right\} \qquad i=1,2......n \qquad \begin{array}{l}(2.2),\\(2.3)\\(2.4)\end{array}$$

Where $\alpha_i=x_i+\frac{1}{3}h_i$, $\beta_i=x_i+\frac{1}{2}h_i$, $\gamma_i=x_i+\frac{2}{3}h_i$

In order to investigate Problem 2.1 we consider a quartic Polynomial q(z) on [0, 1] given by

$$q(z)=q\left(\frac{1}{2}\right)P_1(z)+q'\left(\frac{1}{3}\right)P_2(z)+q'(2/3)P_3(7)+q(0)P_4(z)+q(1)P_5(z) \qquad (2.5)$$

Where

$$P_1(z)=\frac{z}{7}\left[64-208z+288z^2-144z^3\right]$$

$$P_2(z)=\frac{z}{7}\left[-24+99z-129z^2+54z^3\right]$$

$$P_3(z)=\frac{z}{7}\left[3-36z+87z^2-54z^3\right]$$

$$P_4(z)=\left[1-\frac{168z}{7}+\frac{419z^2}{7}-\frac{186z^3}{7}-\frac{72z^4}{7}\right]$$

$$P_5(z)=\frac{z}{7}\left[-4+41z-102z^2+72z^3\right]$$

We are now set to answer problem (2.1) in following theorem.

THEOREM 2.1 :

If $h_{i-1}\geq\ h_i$ there exists unique deficient quartic spline S in S*(4,2,P) which satisfies the interpolatory conditions (2.2) - (2.4) and boundary condition (2.1).

2.2 PROOF OF THE THEOREM

Let $t=\dfrac{x-x_{1-i}}{h_i}, 0\leq t\leq 1$ then in view of condition (2.1) - (2.4).

We now express (2.5) in terms of restriction s_i of s to $[x_{1-i}, x_i]$ as follows :-

$$s_i(x)=f(\beta_i)P_1(t)+h_i f'(\alpha_i)P_2(t)+h_i f'(\gamma_i)P_3(t)+s(x_{1-i})P_4(t)+s(x_i)P_5(t) \quad (2.6)$$

which clearly satisfies (2.1) - (2.4) and $s_i(x)$ is a quartic in $[x_{1-i}, x_i]$ for i=1.....n

Since S is first time continuously differentiable on [0, 1]. Therefore applying continuity condition of first derivative, we get

$$-176\ s_{i-1}h_i+(60h_i+168h_{i-1})s_i+4s_{i+1}h_{i-1}=F_i \text{ say)} \quad (2.7)$$

Where $F_i=\left[64(h_{i-1}f(\beta_i)+h_i f(\beta_{i-1}))-h_i\,h_{i-1}[24f'(\alpha_i)+3f'(\alpha_{i-1})]\right.$

$\left.+h_i h_{i-1}[3f'(\gamma_i)+24f'(\gamma_{i-1})]\right]$

In order to prove theorem 2.1 we shall show that system of equation (2.7) has unique set of solution.

Since $(172h_{i-1}-116h_i)\geq 0$ if $h_{1-i}\geq h_i$.

Therefore, the coefficient matrix of the system of equations (2.7) is diagonally dominant and hence invertible, this complete the proof of theorem 2.1.

III. ERROR BOUNDS

Following method of Hall and Meyer [2], in this sections, we shall estimate the bounds of error function $e(x)=f(x)-s(x)$ for the spline interpolant of theorem 2.1 which are best possible. Let s(x) be the first time continuous differentiable quartic spline function satisfying the conditions of theorem 2.1. Non considering $f\in C^5[0,1]$ and writing M_i [f, x] for the unique quartic which agree with $f(\beta_i), f'(\alpha_i), f'(\gamma_i), f(x_{i-1})$ and $f(x_{i+1})$, we see that for $x\in[x_{i-1}, x_i]$ and we have

$$|e(x)|=|f(x)-s(x)|\cong|f(x)-M_i[f,x]|+|M_i[f,x]-s(x)| \quad (3.1)$$

In order to obtain the bounds of e(x), are proceed to get pointwise bounds of the both the terms on the right hand side of (3.1). The estimate of the first term can be obtained by following a well known theorem of Cauchy [7] i.e.

$$\left|M_i[f,x]-f(x)\right|\leq\frac{h_i^5}{5!}\left|t(1-t)(t-\frac{1}{2})(t-\frac{1}{3})(t-\frac{2}{3})\right|F$$

Where $t=\dfrac{x-x_{1-i}}{h_{i-1}}$, and $U=\max\limits_{0\leq x\leq 1}\left|f^{(5)}(x)\right|$ (3.2)

To get the bounds of $\left|s_l(x)-M_i[f,x]\right|$, we have from (2.6).

$$S_l(x)-M_i[f,x]=[e(x_{i-1})P_4(t)+e(x_i)P_5(t)] \quad (3.3)$$

Thus $\left|s_i(x)-M_i[f,x]\right|\leq\left|e(x_{i-1})\right|\left|P_4(t)\right|+\left|e(x_i)\right|\left|P_5(t)\right|$ (3.4)

Let the $\max\limits_{0\leq i\leq n}\left|e(x_i)\right|$ exist for i=j

Then (3.4) may be written as

$$\left|s_i(x)-M_i[f,x]\right|\leq|e(x_j)|\{|P_4(t)|+|P_5(t)|\} \quad (3.5)$$

Now $\left|P_4(t)\right|+\left|P_5(t)\right|=\left|1-2t\right|\left\{(1-t)\left|7-147t-36t^2\right|+t\left|-4+33t-36t^2\right|\right\}=$

k(t) Say (3.6)

$$\Rightarrow\left|s_i(x)-M_i[f,x]\right|\leq\left|e(x_j)\right|\left|k(t)\right| \quad (3.7)$$

Now we proceed to obtain bound of $\left|e(x_j)\right|$

Replacing $S(x_i)\,by\,e(x_i)$ in (2.7),

We have

$$176h_i e_{i-1}+(60h_i+168h_{i-1})e_i+4e_{i+1}h_{i-1}=F_i+176h_i f_{i-1}-(60h_i+168h_{i-1})f_i$$

$$-4f_{i+1}h_{i-1}=E(f)$$ Say where F_i define in (2.7)

(3.8)

In view of that E(f) is a linear functional which is zero for Polynomial of degree 4 or less, we can apply Peano theorem [7] to obtain

$$E(f)=\int_{x_{j-1}}^{x_{j+1}}\frac{f^5(y)}{4!}E\left[(x-y)_t^4\right]dy \quad (3.9)$$

Thus from (3.9) we have

$$\left|E(f)\right|\leq\frac{1}{4!}F\int_{x_{j-1}}^{x_{j+1}}|E(x-y)_t^4]dy \quad (3.10)$$

Further it can be observed from (3.9) that for $x_{i-1}\leq y\leq x_{i+1}$

$E[(x-y)_t^4$

$$=64[h_{i-1}(\beta_i - y)_+^4 + h_i(\beta_{i-1} - y)_+^4 - h_i h_{i-1}[96(\alpha_i - y)_+{}^3 + 12(\alpha_i - y)_+{}^3]$$

$$+ h_i h_{i-1}[12(\gamma_i - y)_+{}^3 + 48(\gamma_{i-1} - y)_+{}^3] - (60h_i + 168h_{i-1})(x_i - y)_+^4$$

$$-4h_{i-1}(x_{i+1} - y)^4 \qquad (3.11)$$

Rewriting the above expression in the following symmetric form about x_j, we get

$$\Rightarrow -4h_{i-1}[x_i + h_i - y]^4$$

$\gamma_i \le y \le x_{i+1}$

$$\Rightarrow -4h_{i-1}\left[(x_i - y)^4 + h_i(x_i - y)^3 - \frac{5}{3}h_i^4\right]$$

$\beta_i \le y \le \gamma_i$

$$\Rightarrow 4h_{i-1}[15(x_i - y)^4 + 26h_i(x_i - y)^3$$

$$+24(x_i - y)^2 h_i^2 + 8(x_i - y)h_i^3 + \frac{8}{3}h_i^4]$$

$\alpha_i \le y \le \beta_i$

$$\Rightarrow 4h_{i-1}[15(x_i - y)^4 + 7h_i(x_i - y)^3]$$

$x_i \le y \le \alpha_i$

$$\Rightarrow -4h_i[15(x_i - y)^4 - 7h_{i-1}(x_i - y)^3$$

$\gamma_{i-1} \le y \le x_i$

$$\Rightarrow -4h_i[15(x_i - y)^4 - 26h_{i-1}(x_i - y)^3 + 24(x_i - y)^2 h_{i-1}^2$$

$$-8(x_i - y)h_{i-1}^3 + \frac{8}{3}h_{i-1}^4]$$

$\beta_{i-1} \le y \le \gamma_{i-1}$

$$\Rightarrow 4h_i[(x_i - y)^4 - h_{i-1}(x_i - y)^3 - \frac{5}{3}h_{i-1}^4]$$

$\alpha_{i-1} \le y \le \beta_{i-1}$

$$\Rightarrow 4h_i[(x_i - y - h_{i-1}]^4$$

$x_{i-1} \le y \le \alpha_{i-1}$ (3.12)

Thus it is clear from above expression that $E(x-y)_+^4]$ is non-negative for

$x_{i-1} \le y \le x_{i+1}$

Therefore it follows that

$$\int_{x_{i-1}}^{x_{i+1}} \left|E(x-y)_+^4\right| dy = \frac{18.87}{5} h_i h_{i-1}\left[h_{i-1}^4 + h_i^4\right] \quad (3.13)$$

$$\Rightarrow \left|E(f)\right| \le \frac{18.87 h_i h_{i-1}\left[h_{i-1}^4 + h_i^4\right]}{(5!)} \quad (3.14)$$

Thus from (3.8) and (3.14), it follow that -

$$\left|e(x_i)\right| \le \frac{18.87}{5!} \frac{h_i h_{i-1}\left[h_{i-1}^4 + h_i^4\right]}{(172 h_{i-1} - 116 h_i)} F \quad (3.15)$$

Now using (3.2), (3.7) along with (3.15) in (3.1), we have

$$\left|e(x)\right| \le \frac{h^5}{18(5!)}\left[t(1-t)(1-2t)(1-3t)(2-3t)\right]F + \frac{18.87 h^5}{56(5!)} k(t) F \quad (3.16)$$

$$\Rightarrow \left|e(x)\right| \le \frac{h^5}{5!} F\, c(t) \quad (3.17)$$

Where $c(t) = \left|1-2t\right|\left[\frac{t(1-t)\,|1-3t\,\|\,2-3t\,|}{18}\right.$

$\left. + \frac{18.87}{56}\left\{(1-t)(7-147t-36t^2) + t(-4+33t-36t^2)\right\}\right]$

= k*(t) (say)

Thus we prove the following theorem.

3.1 THEOREM : Let s(x) be the quartic spline interpolation of theorem 2.1 interpolating a given function and $f \in C^{(5)}[0,1]$ then

$$|\,e(x) \le k^*(t) \frac{h^5}{5!} \max_{0\le n\le 1} |\,f^5(x)| \quad (3.18)$$

Also, we have

$$|\,e(x_i)\,| \le K_1 \frac{h^{5^1}}{5!} \max_{0\le n\le 1}\left|f^5(x)\right| \quad (3.19)$$

Where $K_1 = \frac{18.87}{56}$

Further more, it can be seen easily that k* (t) in (3.18) can not be improved for an equally spaced partition. In equality (3.19) is also best possible. Also equation (3.16) proves (3.18) whereas (3.19) is direct consequence of (3.15).

Now, we turn to see that inequality (3.19) is best possible in limit case.

Considering $f(x) = \frac{x^5}{5!}$ and using Cauchy formula [7], we have

$$M_i\left[\frac{t^5}{5!}, x\right] - \frac{h^5}{5!} = \frac{h^5}{5!}(1-t)\,t\,(t-\frac{1}{3})(t-\frac{2}{3})(t-\frac{1}{2}) \quad (3.20)$$

Moreover, for the function under consideration (3.8) the following relations holds for equally spaced knots.

$$E(\frac{x^5}{5!})=\frac{18.87}{5!}h^5=-176e_{i-1}+228e_i+4e_{i+1} \quad (3.21)$$

Consider for a moment

$$e_{i-1}=e_i=\frac{18.87h^5}{56(5!)}=e_{1+i} \quad (3.22)$$

we have from (3.5).

$$s(x)-M_i[f,x] \quad (3.23)$$

$$=\frac{18.87}{56}\frac{h^5}{5!}(Q_4(t)+Q_5(t))$$

$$=\frac{18.87}{56}\frac{h^5}{5!}(1-2t)(7-158t+144t^2)$$

$$s(x)-f(x)=\frac{18.87}{56}\frac{h^5}{5!}\{(1-2t)(7-158t+144t^2)\}+\frac{h^5}{5!}t(1-t)(t-\frac{1}{2})(t-\frac{2}{3})(t-\frac{1}{3})\}$$

$$=\frac{h_i^5}{5!}[(\frac{18.87}{56})(1-2t)(7-158t+144t^2)\ +t(1-t)(t-\frac{1}{2})(t-\frac{2}{3})(t-\frac{1}{3})] \quad (3.24)$$

from (3.24), it is clear that (3.16) is best possible. Provided we could prove that

$$e_{1-i}=e_i=e_{1+i}=\frac{18.87}{56}\frac{h^5}{(5!)} \quad (3.25)$$

In fact (3.25) is attained only in the limit. The difficulty will appear in case of boundary condition i.e. $e(x_0)=e(x_n)$ = 0. However it can be shown that as we move many subinterval away from the boundaries.

$$e(x_1)\to\frac{18.87}{56}\frac{h^5}{5!}$$

For that we shall apply (3.21) inductively to move away from the end conditions $e(x_0)=e(x_n)=0$.

The first step in this direction is to establish that $e(x_i)\geq 0$ for some i, 1=1, 2,....n which can be shown by contradictory result. Let $e(x_i)\leq 0$ for some i=1,2,.....n-1.

Now we rewrite equation (3.8)

$$-176e_{i-1}+228e_i+4e_{i+1}=18.87\frac{h^5}{5!} \quad (3.26)$$

$$\frac{18.87}{56}\frac{h^5}{5!}\geq\max|e(x_i)|\geq\frac{1}{2}\{-176e_{i-1}+228e_i+4e_{i+1}\} \quad \text{Since R.H.S is negative}$$

$$\geq 18.87\frac{h^5}{5!}$$

$$\Rightarrow 1\geq 56 \quad (3.27)$$

This is a contradiction.

Hence $e(x_i) \geq 0, \; for\, i = 1, 2, ..., n-1$

Now from (3.26), we can write

$$228 e_i = 18.87 \frac{h^5}{5!} + 176 e_{i-1} - 4 e_{i+1}$$

$$\Rightarrow e_i \leq \frac{18.87}{228} \frac{h^5}{5!}$$

Again using (3.27) in (3.21), we get

$$228 e_i \leq \frac{18.87}{5!} h^5 \left[1 + \frac{172}{228} \right]$$

Repeated use of (3.21), we get

$$e_i \leq \frac{18.87}{228(5!)} h^5 \left[1 + \frac{172}{228} + \left(\frac{172}{228} \right)^2 + ... \right] \qquad (3.28)$$

Now it can easily see that the right hand side of (3.28) tending to $\frac{18.87}{56(5!)} h^5$

$$e(x_i) \leq \frac{18.87}{56} \frac{h^5}{5!} \qquad (3.29\ )$$

which verifies the proof of (3.19).

Thus, corresponding to the function $f(x) = \frac{x^5}{5!}$, (3.28) and (3.29) imply $e(x_i) \to \frac{18.87}{56} \frac{h^5}{5!}$ in the limit for equally spaced knots this completes the proof of theorem 3.1.

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